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Finite Residual Groups and its Products

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ABSTRACT: We say that the finite residual J(G) of a group G is the Intersection of all normal subgroups of finite index of G, and G is residually finite if J(G)=1. In this paper, we show that if $G=A_1\dots A_n$ be product of periodic subgroups $A_1,\dots,An$ satisfying the minimal condition on primary subgroups. Then G satisfies the minimal primary subgroups , and $J(G)= J(A_1)J(A_2)\dots J(An)$.

***Keywords*:** residual, factorize,minimal condition, primary subgroup.

INTRODUCTION

In 1955 N. Ito found an impressive and very satisfying theorem for arbitrary factorized groups. He proved that every product of two abelian groups is metabelian. In 1961 O.H.Kegel and H.Wielandt (1958) expressed the most famous theorem whose states the solubility of all finite products of two nilpotent groups.P.M. Cohn (1956) and L.Redei (1950) considered products of cyclic groups, and around (1965) O.H.Kegel looked at linear and locally finite factorized groups.

In 1968 N.F. Sesekin proved that a product of two abelian subgroups with minimal condition satisfies also the minimal condition. N.F.Seskin and B.Amberg independently obtained a similar result for maximal condition around 1972. Moreover, a little later the later proved that a soluble product of two nilpotent subgroups with maximal condition likewise satisfies the minimal condition, and its Fitting subgroup inherits the factorization. B.Amberg, N.S.Chernikov, S.Franciosi, F.de Gioranni, O.H.Kegel, J.C.Lennox, D.J.S.Robinson, J.F.Roseblade, Y.P.Sysak, J.S.Wilson, and D.I.Zaitsev, development the subjects of products and factorizable group for soluble and nilpotent groups. In 1989, S.V.Ivanov show that every counable group may be embedded in a product of two Tarski groups. Now in this paper we show that if $A_1,\dots,A_n$ is n periodic subgroups such that satisfying the minimal condition on primary subgroups, and $G=A_1\dots A_n$,then G is also satisfies the minimal condition on primary subgroups, and $J(G)=J(A_1)\dots J(A_n)$, where J(G) is the finite residual of group G.

2. Elementary Definitions and Theorems:

In this section we give the elementary definition and Basic theorems whose used the proof of main theorem.

2.1. Definition:

A group G is the product of two subgroups A and B if $G=AB=\{ab|a\in A,b\in B\}$. In this cas we say that G is factorized by A and B. The factorizer of S in G=AB is denoted by X(S) and defined as following : The intersection X(S) of all factorized subgroup S of G=AB containing the subgroup S is the smallest factorized subgroup of G Containing S .(See [5])

2.2. Definition:

Let P is the property pertaining to subgroups of a group G. then we define the P-radical $\rho(G)$ of G is the subgroup generated by all normal P-subgroups of G.(See [5])

2.3. Lemma :

(See [5]) Let A,B and K be subgroups of a group G such that G=AK=BK and $A\cap K=B\cap K$=1. If K is normal in G, then $\rho(A)K=\rho(B)K$, and so this is a normal subgroup of G .

Proof :

Since $\rho(A)K/K=\rho(G/K)=\rho(B)K/K, hence\ \rho(A)K=\rho(B)K, and$ the proof is complete.

2.4.Lemma:

(See [3])Let the group G=AB be the product of two subgroups A and B. Then :

i) If A and B satisfy the maximal condition on subgroups, then G satisfies the maximal condition on normal subgroups.

ii) If A and B satisfy the minimal condition on subgroups, then G satifies the minimal condition on normal subgroups.

Proof:

(i) Let $(H_n)_{n\in N}$ be an ascending sequence of normal subgroups of G. Then $(A\cap H_n)_{n\in N}$ and $(B\cap H_n)_{n\in N}$ are ascending sequence of subgroups of A and B, respectively . Hence $A\cap H_n=A\cap H_{n+1}$ and $B\cap AH_n=B\cap AH_{n+1}$, for almost all n. It follows that $AH_n=AB\cap AH_n=A(B\cap AH_n)=A(B\cap AH_{n+1})=AH_{n+1}$, and so $H_n=H_n(A\cap H_{n+1})=AH_n\cap H_{n+1}=AH_{n+1}\cap H_{n+1}=H_{n+1}$, for almost all n. Therefore G satisfies the maximal condition on normal subgroups.

The proof of (ii) is similar.

2.5.Lemma:

(See [5]) Let N be a non-periodic locally nilpotent normal subgroup of a group G such that G/N is locally finite. If x is an element of infinite order in N and n is a positive integer, the coset $x(x^n)^G$ has order exactly n in the factor group $G/(x^n)^G$. Moreover, if π is the set of prime divisors of n, the group $x^G/(x^n)^G$ is a π –group.

Proof:

Let m be the least positive integer such that x^m belongs to $(x^n)^G$. Then x^m is in $(x^n)^E$ for some finitely generated subgroup E of G. Clearly $(x^n)^E$ lies in $N\cap < x^n, E>$. Since G/N is locally finite, $N\cap < x^n, E>$ has finite index in $<x^n,E>$ and so is a finitely generated nilpotent group.It follow that $(x^n)^E=<((x^m)^g)^{n/m} | g\in E>$ is a finitely generated infinite nilpotent group, which is generated by the (n/m)th powers of its elements. This is possible only if m=n. As the locally nilpotent group $x^G/(x^n)^G$ is generated by π –elements it is clearly a π –group.

2.6. Lemma:

Let the group G=AB=AK=BK be the product of two subgroups A and B and a locally finite normal subgroup K. If E is a finite subgroup of A such that $\pi(E)\cap\pi(K)$ is empty, then there exists an element a of A such that E^a is contained in $A\cap B$.

Proof:

See Lemma 3.2.2. of [5]

2.7.Lemma:

Let π be a set of primes, and let the locally finite group G=AB=AK=BK be the product of two subgroups A and B and a normal π-subgroup K such that $A\cap K=B\cap K=1$. If A_0 and B_0 are normal $\pi' - subgroups$ of A and B, respectively, such that $A_0 \leq B_0 K$, then $N_{A_0}(A_0\cap B_0)=A_0\cap B_0$

Proof :

See Lemma 3.2.3. of [5]

2.8.Lemma:

(See [5]) Let π be a set of primes, and let the locally finite group G=AB=AK=BK be the product of two subgroups A and B and a normal π -subgroup K such that $A\cap K=B\cap K=1$. If A_0 is a normal $\pi'-sabgroup$ of A, then $B_0=O_{\pi'}(B)\cap A_0K$ is a normal subgroup of B satisfying the following conditions.
i) $A_0K=B_0K$
ii) $N_{A_0}(A_0\cap B_0)=A_0\cap B_0$
iii) If A_0 is a belian or finite, then $A_0=B_0$ is a normal subgroup of G.

Proof :

(i) By Lemma 2.3 we have that $O_{\pi'}(A)K=O_{\pi'}(B)K.$ Then
$A_0K=O_{\pi'}(A)K\cap A_0K=O_{\pi'}(B)K\cap A_0K=(O_{\pi'}(B)\cap A_0K)=B_0K.$
ii) This follows from(i) and lemma 2.7.
iii)It is clearly sufficient to show that A_0 it contained in B_0, since then $B_0=A_0K\cap B_0=A_0.$ If A_0 is abelian, then $A_0\cap B_0=A_0$ by(ii), so that A_0 is contained in B_0. suppose that A_0 is finite. By lemma 2.6. there exists an element a of A such that $A_0=A_0^a\le A\cap B\le N_G(B_0).$ Therefore A_0B_0 is a $\pi'-group.$ Now $A_0B_0=A_0B_0\cap B_0K=B_0(A_0B_0\cap K)=B_0$, so that A_0 is contained in B_0. The lemma is proved.

2.9.Definition :

Recall that a group G is called radicable if for each element x of G and for each positive integer n there exists an element y of G such that $x=y^n$. A group G is reduced if it has no non-trivial radicable subgroups.

2.10.Lemma :

Let G be a locally finite group having no infinite simple sections. then :
i) If G satisfies the minimal condition on p-subgroups for some prime p, then the factor group $G/O_{p'}(G)\,contains$ a Chernikov p-subgroup of finite index;
ii) if G satisfies the minimal condition on p-subgroups for every prime p, then the finite residual J of G is radicable abelian and the sylow subgroups of G/J are finite.

Proof:

See ([8], p.94).

Note:

For see definition of locally finite group refered to" Locally Finite Groups" book of Kegel and Wehrfritz. ([8])

2.11.Theorem:

(See [7],[10]) Let the hyper-((locally nilpotent) or finte) group G=AB be the product of two periodic hyper-(abelian or finite) subgroups A and B. Then the following hold.
(i) G is periodic.
(ii) If the Sylow p-subgroups of A and B are Chernikov (respectively : finite, trivial), then the p-component of every abelian normal section of G is Chernikov (respectively: finite, trivial).

Proof :

(i) Assume that G is not periodic. Without loss of generality We may suppose that G has no non-trivial periodic normal subgroups. Then G contains a non-trivial torsion-free locally nilpotent normal subgroup K. Clearly the factorizer X (K) is also a counterexample, so that we may suppose that G has a triple factorization G=AB=AK=BK, Where $A\cap K=B\cap K=1.$ Since A is hyper-(abelian or finite), it contains a non-trivial normal subgroup N which is either finite or an abelian p-group. As NK is normal G=AK, the intersection $NK\cap B$ is normal in B. If N is contained in B, it is a normal subgroup of $AB=G.$ This contradiction shows that N is not contained in B. Let a be an element of N\B, and write $a=b^{-1}x$, with b in B and x in K. Let q be a prime which is not in the finite set π (N).Consider the normal closures $L=x^G$ and $V=(x^q)^G$. Since x has infinite order, by Lemma.2.5 the coset $\bar{x}=xV$ has order q in $\bar{G}=G/V$, and $\bar{L}=L/V$ is a q-group. The factorizer $\bar{X}$ of $\bar{L}$ in $\bar{G}=\bar{A}\bar{B}$ has the triple factorization

$$\bar{X}=\bar{A}^*\bar{B}^*=\bar{A}^*\bar{L}=\bar{B}^*\bar{L},$$

Where $\bar{A}^*=\bar{A}\cap\bar{B}\bar{L}$ and $\bar{B}^*=\bar{B}\cap\bar{A}\bar{L}$. By Lemma 2.8(iii) the subgroup $\bar{N}\cap\bar{A}^*$ is contained in $\bar{B}^*$, so that $\bar{a}=\bar{b}^{-1}\bar{x}$ is in $\bar{B}^*$. Then $\bar{x}=\bar{b}\bar{a}$ belongs to $\bar{B}\cap\bar{K}=1$. This contradiction proves (i).

(ii) Assume that G contains an abelian normal section M whose p-component is not a Chernikov group. Without loss of generality we may suppose that M is an abelian normal p-subgroup of G. As the factorizer X(M) is also a counterexample, it can also be assumed that G has a triple factorization G=AB=AM=BM. Since $A\cap M$ and $B\cap M$ are Cherinkov normal p-subgroups of G, we may replace G by G/($A\cap M$)($B\cap M$) and hence suppose that $A\cap M$ = $B\cap M$ =1.

By Lemma 2.3 the subgroup $O_{p'}(A)M=O_{p'}(B)M$ is normal in G. Clearly $O_{p'}$ (G) is contained in H=$O_{p'}$ (A)M. Hence $O_{p'}(G)\le O_{p'}(H)\le O_{p'}(A)$, so that $O_{p'}(G)$ is contained in $A\cap B$. Factoring out $O_{p'}(G)$, we may suppose that G has no non-trivial normal p'-subgroups, and $A\cap M$ = $B\cap M$ =1. If $O_{p'}(A)$ is not trivial, it contains a non-trivial normal subgroup A_0 of A which is either finite or abelian. Then A_0 is normal in G by Lemma 2.8(iii). This contradiction shows that $O_{p'}(A)=O_{p'}(B)=1$ .Since A and B satisfy the minimal condition on p-subgroups, it follows from Lemma 2.10(i) that A and B contain normal p-subgroups of finite index. In particular, A and B are Cherinkov groups. The group G=AM also contains a normal p-subgroup P of finite index. Moreover, G satisfies the minimal condition on normal subgroups, by Lemma 2.4(ii). Then Also P has the minimal condition on normal subgroups and hence is a Chernikov group (Robinson 1972, Part 1, Theorem 5.21 and Corollary 2 to Theorem 5.27). Therefore G is a Cherinkov group. This contradiction completes the proof of the first statement of (ii). The proofs of the other two statements are similar.

Note :

For see definition of Chernikov group refered to [8].

2.12.Theorem:

(See [6]) Let the soluble- by- finite group G=AB be the product of two periodic subgroups A and B satisfying the minimal condition of p-subgroups for some prime p. Then G satisfies the minimal condition on p-subgroups.

Proof: By theorem 2.11(ii) every abelian normal section of G satisfies the minimal condition on p-subgroups. Since this property is obviously inherited by extensions, G has the minimal condition on p-subgroups. And the proof is complete.

3. Main result :

In this section by use of Lemmas and Theorems of section 2, we express and proved the main theorem of paper as following.

3.1. Main Theorem:

Let the soluble-by-finite group G=AB be the product of n periodic subgroups A_1,…, and A_n satisfying the minimal condition on primary subgroups. Then G satisfies the minimal condition on primary subgroups, and $J(G)=J(A_1)\ldots J(A_n)$, where J(G) is the finite residual of G.

Proof:

The first we proved that if A and B be two periodic subgroups that satisfying the minimal condition on primary subgroup and G=AB is soluble-by-finite group. Then G is also satisfies the minimal condition on primary subgroups, and J(G)=J(A)J(B). For this we say that by used from Theorem 2.11 and Theorem 2.12 follow that G is a locally finite group eith the minimal condition on primary subgroups. Then by Lemma 2.10(ii) the finite residual J=J(G) is a radicable abelian group whose primary components J_p satisfy the minimal condition, and the factor group G/J has finite Sylow subgroups. It is enough to prove that $J_p=J_p(A)J_p(B)$, for every prime p.

The factorizer X of J_p has the triple factorization X =A*B*=A*J_p=B*J_p, Where A*= $A \cap BJ_p$ and B*= $B \cap BJ_p$. The subgroup $J_p(A)$ is contaned in J_p, and so it is normal in X=A*J_p. Similarly $J_p(B)$ is normal in X. Therefore N=$J_p(A)J_p(B)$ is a normal subgroup of X. As the factor groups A/$J_p(A)$ and B/$J_p(B)$ have finite Sylow p-subgroups, by Theorem 2.11(ii) the Sylow p-subgroups of X/N are also finite. In particular J_p/N is finite, so that Jp=N=$J_p(A)J_p(B)$. Now by induction on n, follows that if G=$A_1 \ldots A_n$, whose $A_1,\ldots,A_n$ are n periodic subgroups that satisfying the minimal condition on primary, then G is also satisfies the minimal condition primary subgroups, andn $J(G) = J(A_1) \ldots J(A_n)$, and the proof of Theorem is complete.

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